Math 255A' Lecture 13 Notes

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October 25, 2019

1 The Krein-Milman Theorem and The Markov-Kakutani Fixed Point Theorem

Today's lecture was given by a guest lecturer, Professor Dimitri Shlyakhtenko.

1.1 The Krein-Milman theorem

Definition 1.1. Let $K \subseteq X$ be convex. Then a is an **extreme point** of K if $a \in K$ and if whenever $a = \alpha + (1-\alpha)y$ for some $\alpha \in [0,1]$ and $x, y \in K$, then $\alpha = 0$ or $\alpha = 1$.

So extreme points cannot be on the interior of a line segment in K. The set of extreme points is denoted as $\operatorname{ext}(K)$.

Example 1.1. Suppose $K = \{f \in L^1([0,1]) : \|f\|_1 \leq 1\}$. What are the extreme points of K? If $\|f\|_1 = 1$, then $\int_0^1 |f(t)|\,dt = 1$. The primitive $F(T) = \int_0^T |f(t)|\,dt$ is continuous, so there is a T such that $\int_0^T |f(T)|\,dt = 1/2$. Now define

$$h(t) = \begin{cases} 2f(T) & t \leq T \\ 0 & \text{otherwise}, \end{cases} \qquad g(t) = \begin{cases} 0 & t \leq T \\ 2f(t) & \text{otherwise}. \end{cases}$$

Then $\|h\|_2 = \|g\|_2 = 1$, and $f = \frac{1}{2}h + \frac{1}{2}g$. So there are no extreme points.

Theorem 1.1 (Krein-Milman)**.** *Let X be an LCS, and let K be a nonempty, compact, convex subset. Then $K = \overline{\operatorname{co}}(\operatorname{ext} K)$. In particular, $\operatorname{ext} K \neq \varnothing$.*

Corollary 1.1. *If $B \subseteq X$ is a nonempty, convex subset such that $\operatorname{ext}(B) \neq \varnothing$, then no LCS structure on X makes B compact.*

Corollary 1.2. *If X is a Banach space and $\operatorname{ext}(X)_1 = \varnothing$, then $X \neq Y*$ for any Y.*

Example 1.2. This shows that L^1 is not the dual of anything.

Proposition 1.1. *Let $K \subseteq X$ be convex. The following are equivalent:*

1. $a \in \operatorname{ext} K$.

2. *If* $a = \frac{1}{2}(x_1 + x_2)$ *with* $x_1, x_2 \in K$, *then* $x_1 = x_2$.

3. *If* $x_1, \ldots, x_k \in K$ *and* $a \in \operatorname{co}\{x_1, \ldots, x_k\}$, *then* $a = x_j$ *for some* j.

4. $K \setminus \{a\}$ *is convex.*

Here is the idea of the proof of the Krein-Milman theorem: Look for maximal (nontrivial) relatively open convex subsets (and hope that these are the same as $\{K \setminus \{a\} : a \in \operatorname{ext} K\}$).

Proof. We want to use Zorn's lemma. Let $\mathcal{U} = \{U \subseteq K : U \text{ rel. open, convex}, U \neq \varnothing, U \neq K\}$. This is nonempty and ordered by inclusion. Assume that $\mathcal{U}_0 \subseteq \mathcal{U}$ is a chain. Let $U_0 = \bigcup_{U \in \mathcal{U}_0} U$; this is open (as a union of open sets), and it is convex.[1] U_0 is nontrivial, as well: if $U_0 = K$, then $\mathcal{U}_0$ is an open cover for K, which means that $K \subseteq U$ for some $U \in \mathcal{U}_0$, This is a contradiction.

By Zorn's lemma there exists a maximal element $U \in \mathcal{U}$. Let $x \in L$ and $\lambda \in [0,1]$. Define $T_{x,\lambda} : K \to K$ by $T_{x,\lambda}(y) = \lambda y + (1-\lambda)x$. This is continuous and **affine** (i.e. $T_{x,\lambda}(\sum_j \alpha_j y_j) = \sum_j \alpha_j T(y_j)$ if $\alpha_j \geq 0$ and $\sum_j \alpha_j = 1$).

We claim that if $\lambda < 1$ and $x \in U$, then $T_{x,\lambda}(U) \subseteq U$. Thus, $U \subseteq T_{x,\lambda}^{-1}(U)$, which is an open, convex set. If $y \in \overline{U} \setminus U$, then $T_{x,\lambda}(y) \in [x, y) \subseteq U$. So if $\overline{U} \subseteq T_{x,\lambda}^{-1}(U)$, then $T_{x,\lambda}^{-1}(U) = K$. Thus, $T_{x,\lambda}(K) \subseteq U$ for all $x \in U$ and $\lambda \in [0,1)$.

We claim that if $V \subseteq K$ is open and convex, then $V \cup U = U$ or $V \cup U = K$. This is because $V \cup U$ is open, and the conclusion above implies that $V \cup U$ is convex. If $V \cup U \neq K$, then $V \cup U \subseteq U$ by maximality.

We now claim that $K \setminus U$ is one point. If $a, b \in K \setminus U$ and $a \neq b$, then choose disjoint, open, convex subsets $V_a, V_b \subseteq K$ with $a \in V_a, b \in V_b$. Then $V_a \cup U \neq U$, so $V_a \cup U = K$. However, this implies $b \in V_a \cap V_b$, which gives a contradiction.

We now claim that if $V \subseteq X$ is open, convex, and $\operatorname{ext} K \subseteq V$, then $K \subseteq V$: Suppose not, so there exists an open, convex $V \subseteq X$ such that $\operatorname{ext} K \subseteq V$ by $V \cap K \neq K$. Then $V \cap K \subseteq \mathcal{U}$, so there is a maximal $U \in \mathcal{U}$ such that $V \cap K \subseteq U = K \setminus \{a\}$ and $a \in \operatorname{ext}(K)$. Then $a \notin V$, which is a contradiction.

To finish the proof: Let $E = \overline{\operatorname{co}}(\operatorname{ext} K)$. If $x^* \in X^*$, $\alpha \in \mathbb{R}$, and $E \subseteq \{x \in X : \operatorname{Re}\langle x, x^* \rangle < \alpha\} = V$, then $K \subseteq V$. Hahn-Banach says that E is the intersection of such sets V. So $E \supseteq K$. ◻

Here is another theorem. This is

Theorem 1.2. *Let* X *be an LCS, and let* $X \subseteq K$ *be compact, and convex. Assume that* $F \subseteq K$ *is such that* $K = \overline{\operatorname{co}}(F)$. *Then* $\operatorname{ext}(K) \subseteq \overline{F}$.

[1] It also has a hilarious notation.

1.2 The Markov-Kakutani fixed point theorem

Fixed point theorems allow us to show the existence of desired objects by expressing them as a fixed point of some map(s).

Theorem 1.3 (Markov-Kakutani fixed point theorem)**.** *Let $K \subseteq X$ be a nonempty, compact, convex set. Let $\mathcal{F}$ be a family of affine maps $K \to K$ which is* ***abelian*** *($ST = TS$ for all $S, T \in \mathcal{F}$). Then there exists a fixed point $x_0 \in K$ such that $T(x_0) = x_0$ for all $T \in \mathcal{F}$.*

Proof. Let $T \in \mathcal{F}$. Define $T^{(n)} = \frac{1}{n}\sum_{k=0}^{n-1} T^k$. Then $T^{(n)}$ is a again an affine map taking $K \to K$. If $S, T \in \mathcal{F}$, then $S^{(n)}, T^{(m)}$ commute for all n, m. Let $\mathcal{K} = \{T^{(n)}(K) : T \in \mathcal{F}, n \geq 1\}$, which is a collection of compact, convex sets. If $T_1, \dots, T_p \in \mathcal{F}$ and $n_1, \dots, n_p \geq 1$, then

$$T_1^{(n_1)} \circ \cdots \circ T_p^{(n_p)}(K) \subseteq \bigcap_{j=1}^{p} T_j^{(n_j)}(K).$$

These are arbitrary elements of $\mathcal{K}$, then $\mathcal{K}$ has the finite intersection property. So there exists an $x_0 \in \bigcap_{K' \in \mathcal{K}} K'$.

We claim that x_0 is the desired fixed point. Take $t \in \mathcal{F}$, and let $n \geq 1$. Then $x_0 \in T^{(n)}(K)$, so $x_0 = T^{(n)}(x)$ for some x. In particular,

$$x_0 = \frac{1}{n}x + T(x) + \cdots + T^{n-1}(x)).$$

Applying T, we get

$$T(x_0) = \frac{1}{n}(T(x) + \cdots + T^{n-1}(x) + T^n(x)).$$

Subtracting this, we get

$$T(x_0) - x_0 = \frac{1}{n}(T^n(x) - x) \in \frac{1}{n}(K - K),$$

where $K - K$ is compact. This is true for any n. If U is an open neighborhood of 0, then there exists some n such that $\frac{1}{n}(K - K) \subseteq U$. Then $T(x_0) - x_0 \in U$ for all open neighborhoods U of 0, so $T(x_0) = x_0$. $\square$